# On the Counter Collision Probability of GCM*

Keisuke Ohashi, Nagoya University
Yuichi Niwa, Nagoya University
Tetsu Iwata, Nagoya University

Early Symmetric Crypto (ESC) seminar
January 14--18, Mondorf-les-Bains, Luxembourg

*Work in Progress

# GCM

- Galois/Counter Mode
- authenticated encryption mode of 128-bit blockciphers
- designed by McGrew and Viega in 2004 [MV04]
- selected as the NIST recommended authenticated encryption mode in 2007
- widely used in practice
  - ISO/IEC 19772, IEEE P1619.1, NSA Suite B, IETF IPsec, SSH, SSL,…

[MV04] David A. McGrew and John Viega: The Security and Performance of the Galois/Counter Mode (GCM) of Operation. INDOCRYPT 2004. Full version in Cryptology ePrint Archive: Report 2004/193

# Overview

- $\mathbf{Adv}^{\mathrm{priv}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \dfrac{0.5(\sigma+q+1)^2}{2^n} + \dfrac{2^{22}q(\sigma+q)(\ell_N+1)}{2^n}$
- $\mathbf{Adv}^{\mathrm{auth}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \dfrac{0.5(\sigma+q+q'+1)^2}{2^n} + \dfrac{2^{22}(q+q'+1)(\sigma+q)(\ell_N+1)}{2^n} + \dfrac{q'(\ell_A+1)}{2^\tau}$
- "big constant"
- Joux at Dagstuhl Seminar (January 2012): Do you have an attack that matches the bound (exploiting the fact that there is a big constant)? --- I don't know
  - tightness of the bounds, possibility of improvement

[IOM12] Tetsu Iwata, Keisuke Ohashi, and Kazuhiko Minematsu : Breaking and Repairing GCM Security Proofs. CRYPTO 2012. Full version in Cryptology ePrint Archive: Report 2012/438

# Overview

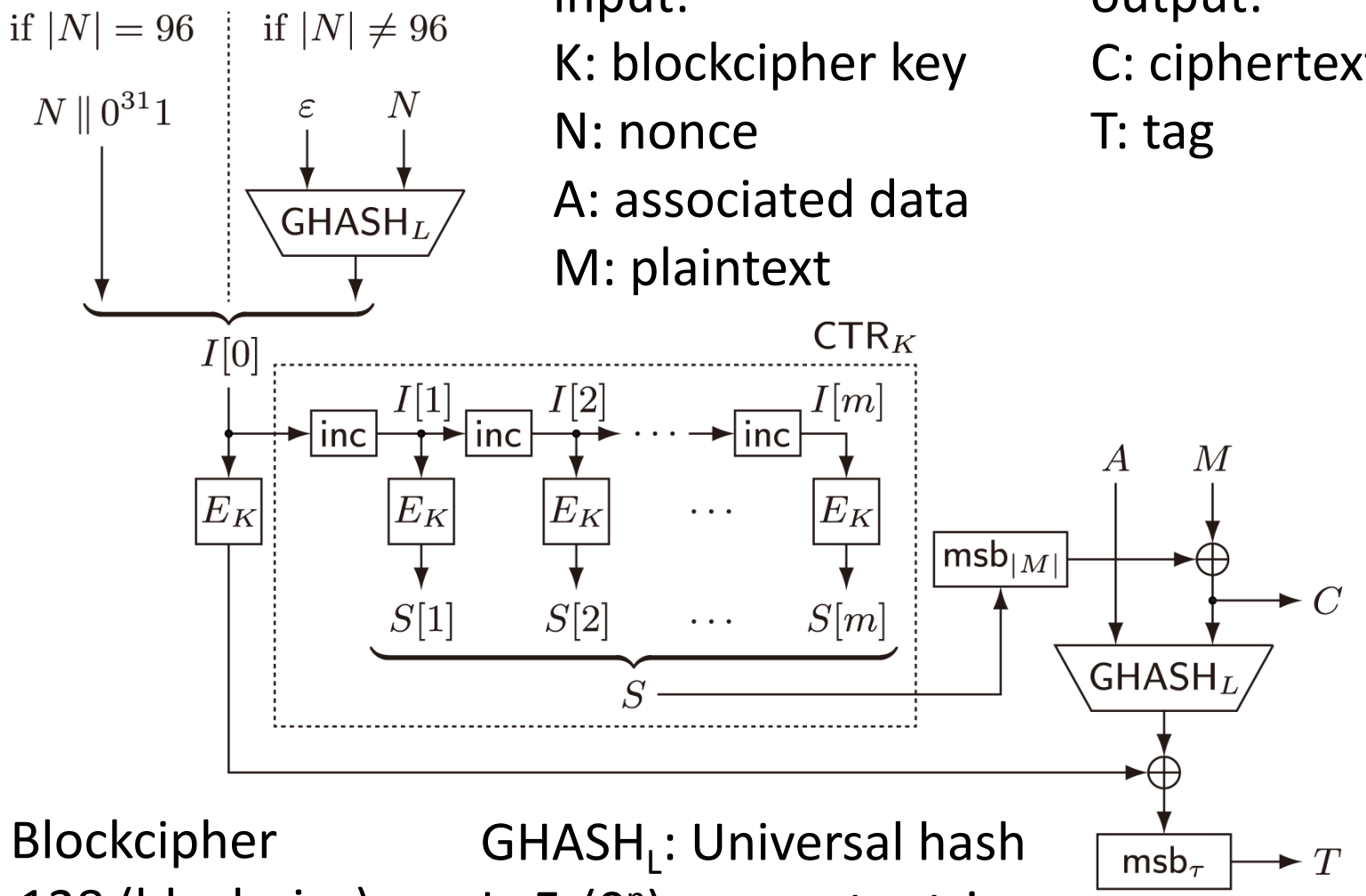
- $\mathbf{Adv}^{\mathrm{priv}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \dfrac{0.5(\sigma+q+1)^2}{2^n} + \dfrac{2^{22}q(\sigma+q)(\ell_N+1)}{2^n}$
- $\mathbf{Adv}^{\mathrm{auth}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \dfrac{0.5(\sigma+q+q'+1)^2}{2^n} + \dfrac{2^{22}(q+q'+1)(\sigma+q)(\ell_N+1)}{2^n} + \dfrac{q'(\ell_A+1)}{2^\tau}$
- ESC (January 2013): I still don't know, but we have made some progress

[IOM12] Tetsu Iwata, Keisuke Ohashi, and Kazuhiko Minematsu : Breaking and Repairing GCM Security Proofs. CRYPTO 2012. Full version in Cryptology ePrint Archive: Report 2012/438

# Encryption Algorithm of GCM

input:
K: blockcipher key
N: nonce
A: associated data
M: plaintext

output:
C: ciphertext
T: tag




$E_K$: Blockcipher
n = 128 (block size)

$\text{GHASH}_L$: Universal hash
$L=E_K(0^n)$, ε: empty string

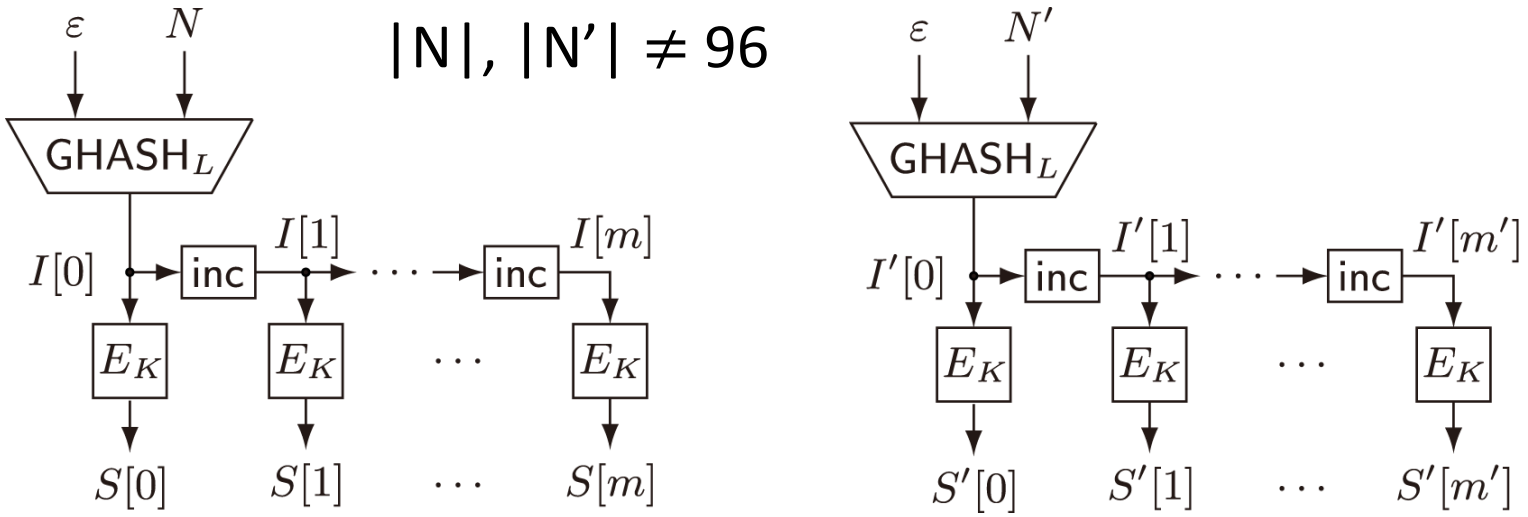
# Increment Function in GCM

- $inc( X \,||\, Y ) = X \,||\, (Y+1 \bmod 2^{32})$
  - $|X| = 96$, $|Y|=32$
  - Example
    - $inc( 0x0…01 ) = 0x0…02$
    - $inc( 0x0…0ffffffff ) = 0x0…0$
- $inc^r ( Z )$: apply $inc( \cdot )$ on Z for r times
  - $|Z| = 128$

# $GHASH_L(\varepsilon, N)$

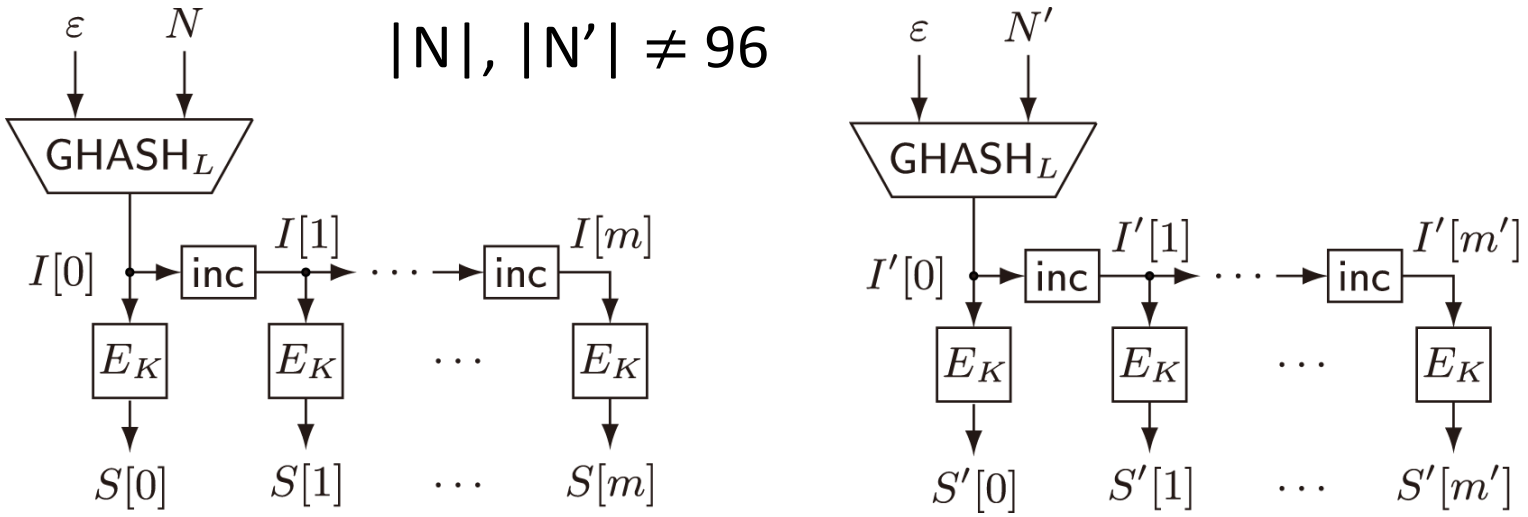
- A universal hash function defined over $GF(2^{128})$, which is defined by the irreducible polynomial $p(x) = 1+x+x^2+x^7+x^{128}$, where the multiplicative identity element is 0x80…0
- $N \,||\, 0…0 \,||\, |N|_{128} = ( X[1],…,X[x] )$
- $GHASH_L(\varepsilon, N) = X[1] \cdot L^x \oplus X[2] \cdot L^{x-1} \oplus … \oplus X[x] \cdot L$
- Example
  - N = 0x00000000 00000000 02 (72 bits)
  - $GHASH_L(\varepsilon, N)$
    $= 0x00000000\ 00000000\ 02000000\ 00000000 \cdot L^2$
    $\oplus\ 0x00000000\ 00000000\ 00000000\ 00000048 \cdot L$
  - if $|N| \leq 128$ then $\deg( GHASH_L(\varepsilon, N) ) \leq 2$

# Counter Collision




- A counter collision: for some r,
  - I[r] = I'[0]
  - $\text{inc}^r(\text{GHASH}_L(\varepsilon, N)) = \text{GHASH}_L(\varepsilon, N')$
  - $\text{Coll}_L(r, N, N')$

# Counter Collision




- A counter collision is a bad event: I[1] = I'[0], I[2] = I'[1], …
  - xor of two ciphertexts = xor of two plaintexts
  - the information about plaintexts is leaked
- We need to show that $Pr_L[\ Coll_L(r, N, N')\ ]$ is small

# $Pr_L[\ Coll_L(r, N, N')\ ]$ Is Small

- [Lemma 3, MV04]

$$Pr_L[\ Coll_L(r, N, N')\ ] \leq \max\{\ d, d'\ \} / 2^{128}$$

  where $d = deg(\ GHASH_L(\varepsilon, N)\ )$, $d' = deg(\ GHASH_L(\varepsilon, N')\ )$
- turns out to be wrong for some ( r, N, N' ) [IOM12]
  - r = 0x0…01, N = 0x0…02 (72 bits), N' = 0x0…06 (72 bits)
  - [Lemma 3, MV04] says $Pr_L[\ Coll_L(r, N, N')\ ] \leq 2 / 2^{128}$
  - but $Pr_L[\ Coll_L(r, N, N')\ ] \geq 32 / 2^{128}$ (a lower bound)
    - a distinguishing attack with $Adv^{priv}_{GCM[Rand(n),\tau]}(A) \geq 32/2^{128}$

# $Pr_L[\ Coll_L(r, N, N')\ ]$ Is Small

- [Lemma 2, IOM12] For each $0 \leq r \leq 2^{32}-1$

$$Pr_L[\ Coll_L(r, N, N')\ ] \leq \alpha_r \max\{ d, d' \} / 2^{128}$$

where $d = \deg(\ GHASH_L(\varepsilon, N)\ )$, $d' = \deg(\ GHASH_L(\varepsilon, N')\ )$

- $\alpha_r$ can be large
  - $\alpha_r = 32$ when r = 0x0…01
  - $\alpha_r = 3524578$ when r = 0x2aaaaaab, 0x55555555, 0xaaaaaaab, 0xd5555555
    - 3524578 is about $2^{22}$
    - "big constant" appears in the upper bound

# Dagstuhl Seminar (January 2012)

- Joux: Do you have an attack that matches the bound (exploiting the fact that there is a big constant)?
  - finding (r, N, N') such that

$$\Pr_L[\ \mathrm{Coll}_L(r, N, N')\ ] \geq (\text{big constant}) / 2^{128}$$

# Examples in [IOM12]

- r = 0x0…01, N = $0x0^{17}2$, N’ = $0x0^{17}6$ (72 bits)
- r = 0x0…01, N = $0x0^{15}20^{12}$, N’ = $0x0^{15}60^{12}$ (112 bits)
- r = 0x0…01, N = $0x0^{17}20^{10}$, N’ = $0x0^{17}60^{10}$ (112 bits)
- r = 0x0…01, N = $0x0^{14}40^{3}$, N’ = $0x0^{14}c0^{3}$ (72 bits)

  - $Pr_L[\ Coll_L(r, N, N')\ ] \geq 32 / 2^{128}$

# How We Found

- $|N|, |N'| \leq 128$
- $GHASH_L(\varepsilon, N) = (N \,||\, 0\ldots0) \cdot L^2 \oplus |N|_{128} \cdot L = U \cdot L^2 \oplus V \cdot L$
- $Pr[\, inc^r(\, GHASH_L(\varepsilon, N)\,) = GHASH_L(\varepsilon, N')\,]$
- $inc^1(\, U \cdot L^2 \oplus V \cdot L\,) = U' \cdot L^2 \oplus V' \cdot L$
  - started with random ( U, V, U', V' )
  - at some point we found that ( U, V, U', V' ) of the form
    - $V = V'$
    - $U = 0^{8i} \,||\, X \,||\, 0^{120-8i}$
    - $U' = 0^{8i} \,||\, X' \,||\, 0^{120-8i}$
      - $|X|, |X'| = 8$

    has many solutions

# Try the Same for r = 0x55555555

- r = 0x55555555
- for each ( U,V,U',V' ) // V=V'

  counter = 0

  for 3524578 values of C

  solve $U \cdot L^2 \oplus V \cdot L \oplus C = U' \cdot L^2 \oplus V \cdot L$

  if $inc^r( U \cdot L^2 \oplus V \cdot L ) = U' \cdot L^2 \oplus V \cdot L$ then counter++

  }

  output ( U,U',V ) if counter is large

# Result

- r = 0x55555555
- counter = 8495 for the following values of (N,N’):
  - (0x0…01d000000000000, 0x0…02b000000000000)
  - (0x0…02c000000000000, 0x0…064000000000000)
  - (0x0…0160000000000, 0x0…0320000000000)
  - (0x0…0270000000000, 0x0…07d0000000000)
  - |N| = |N’| = 112

# So?

- $Pr_L[\ Coll_L(r, N, N')\ ] \geq 8495 / 2^{128}$
  - $Adv^{priv}_{GCM[Rand(n),\tau]}(A) \geq 8495/2^{128}$
- $Pr_L[\ Coll_L(r, N, N')\ ] \geq 4247\ \max\{d, d'\} / 2^{128} \geq 2^{12}\ \max\{d, d'\} / 2^{128}$
- Not as large as $2^{22}$ , but the gap is now smaller
  - 32 vs $2^{22}$ -> $2^{12}$ vs $2^{22}$

# Security Bounds [IOM12]

- $\mathbf{Adv}^{\mathrm{priv}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \dfrac{0.5(\sigma+q+1)^2}{2^n} + \dfrac{2^{22}q(\sigma+q)(\ell_N+1)}{2^n}$
- $\mathbf{Adv}^{\mathrm{auth}}_{\mathrm{GCM}[\mathrm{Perm}(n),\tau]}(\mathcal{A}) \leq \dfrac{0.5(\sigma+q+q'+1)^2}{2^n} + \dfrac{2^{22}(q+q'+1)(\sigma+q)(\ell_N+1)}{2^n} + \dfrac{q'(\ell_A+1)}{2^\tau}$
- The tightness is open
- There is a possibility to reduce $2^{22}$ to a smaller constant, but it cannot be less than $2^{12}$ (if we follow the proof strategy in [IOM12])

# ASK 2012 (August 2012)

- Try to find (r,N,N') that gives a higher collision probability
  - (U,V,U',V') can take approximately $2^{128}$ $2^{128}$ values
- Yasuda: Try smaller GCM?

# Small GCM with n = 16

- block size is n = 16 bits
- inc( · ) operates on 4 bits
- GHASH is defined over GF($2^{16}$) with the lexicographically first irreducible polynomial $p(x) = 1+x+x^3+x^5+x^{16}$

# Small GCM with n = 16

- $Pr_L[\ Coll_L(r, N, N')\ ] \leq \alpha_r \max\{ d, d' \} / 2^{16}$
  - $\alpha_r = 5$ (max) when r = 0x3, 0x5, 0xb, 0xd
- $|N|, |N'| \leq 16$
  - $GHASH_L(\varepsilon, N) = (N \,||\, 0\ldots0) \cdot L^2 \oplus |N|_{16} \cdot L = U \cdot L^2 \oplus V \cdot L$
  - $Pr[\ inc^r(\ GHASH_L(\varepsilon, N)\ ) = GHASH_L(\varepsilon, N')\ ] \leq 10 / 2^{16}$
- $inc^r(\ U \cdot L^2 \oplus V \cdot L\ ) = U' \cdot L^2 \oplus V' \cdot L$
  - also consider $V \neq V'$
  - about $2^{33}$ values of ( U,V,U',V')

# Result

- Pr [ $\text{Coll}_L(r, N, N')$ ] = $10 / 2^{16}$ holds
  - for 87,406 pairs of (N,N') when r = 0x3, 0xd
  - for 86,951 pairs of (N,N') when r = 0x5, 0xb

- For any (r,N,N'), $\text{Pr}_L[\ \text{Coll}_L(r, N, N')\ ] \leq \alpha_r \max\{ d, d' \} / 2^{16}$
- There exists (r,N,N') such that $\text{Pr}_L[\ \text{Coll}_L(r, N, N')\ ] = \alpha_r \max\{ d, d' \} / 2^{16}$
- There is an attack that matches the bound
- The "big constant" in security bounds cannot be replaced by a smaller one

# Small GCM with n = 20

- block size is n = 20 bits
- inc( · ) operates on 5 bits
- GHASH is defined over GF($2^{20}$) with the lexicographically first irreducible polynomial $p(x) = 1+x^3+x^{20}$

# Small GCM with n = 20

- $Pr_L[\ Coll_L(r, N, N')\ ] \leq \alpha_r \max\{ d, d' \} / 2^{20}$
  - $\alpha_r = 8$ (max) when r = 0x5, 0xb, 0x15, 0x1b
- $|N|, |N'| \leq 20$
  - $GHASH_L(\varepsilon, N) = (N \,||\, 0\ldots0) \cdot L^2 \oplus |N|_{20} \cdot L = U \cdot L^2 \oplus V \cdot L$
  - $Pr[\ inc^r(\ GHASH_L(\varepsilon, N)\ ) = GHASH_L(\varepsilon, N')\ ] \leq 16 / 2^{20}$
- $inc^r(\ U \cdot L^2 \oplus V \cdot L\ ) = U' \cdot L^2 \oplus V' \cdot L$
  - also consider $V \neq V'$
  - about $2^{41}$ values of ( U,V,U',V' )
- Result: $Pr[\ Coll_L(r, N, N')\ ] = 16 / 2^{20}$ holds
  - for 49,065 pairs of (N,N') when r = 0x5
  - There is an attack that matches the bound

# Conclusions

- Joux: Do you have an attack that matches the bound?

- The tightness is still open for n = 128, but the gap is now smaller ($2^{12}$ vs $2^{22}$)
- We have a matching attack for small versions of GCM (n = 16, 20)
- Plan: to investigate small versions of GCM for n = 24, 28, 32, ...